XL. *Methodus inveniendi Lineas Curvas ex proprietatibus Variationis Curvaturæ. Auctore* Nicolao Landerbeck, *Mathes. Profess. in Acad. Upsaliensi Adjuncto. Communicated by* Nevil Maskelyne, *D. D. F. R. S. and Astronomer Royal.*

Read July 1, 1784.

PARS SECUNDA*.

CURVAS, ex proprietate variationis curvaturæ invenire, indice per functionem coordinatarum cujusdam expresso, problema etsi indeterminatum est; juvat tamen ad curvas cognoscendas, quum facile et sponte sese offerunt conditiones determinantes qui rei conveniunt et quæ in casu quovis examini subjecto locum habent. Quo consilio et qua arte calculum inire oporteat, ut et hæc et his affinia peragenda sint, quæ ad curvas ex curvaturæ variatione cognoscendas pertineant, per theoremata quæ sequuntur, exponere conabor.

THEOREMA I. (Vide tab. XXI. fig. 2.)

Si curvæ cujusdam LC index variationis curvaturæ sit T, radius curvedinis R, sinus anguli BCD p, posito sinu toto 1, arcus curvæ LC z coordinatæ perpendiculares x et y earumque fluxiones dp, dz, dx, et dy dicantur, erit $\frac{dz}{\int Tdz} = -\frac{dp}{\sqrt{1-p^2}}$.

Quoniam $dx = -Rdp$ et $dz = -\frac{dx}{\sqrt{1-p^2}}$ habetur $\frac{dz}{R} = -\frac{dp}{\sqrt{1-p^2}}$

* See Vol. LXXIII. p. 456.

et

et quum $dR = Tdz$ erit $R = \int Tdz$ et fubftitutione $\frac{dz}{\int Tdz} = -\frac{dp}{\sqrt{1-p^2}}$.

Cor. 1. Hinc obtinetur $\frac{dx}{R} = -dp$, $\frac{dy}{R} = -\frac{pdp}{\sqrt{1-p^2}}$ et $\frac{dz}{R} = -\frac{dp}{\sqrt{1-p^2}}$.

Cor. 2. Si Tangens anguli BCD per r, Secans per s defignentur habetur $\frac{dz}{\int Tdz} = -\frac{dr}{1+r^2}$ et $\frac{dz}{\int Tdz} = -\frac{ds}{s\sqrt{s^2-1}}$.

Schol. 1. Ex hoc theoremate facilis deducitur methodus generaliter calculandi variationem curvaturæ curvæ cujufcumque. Nam $\int Tdz = -\frac{dz\sqrt{1-p^2}}{dp}$, quantitas vero $\frac{dz\sqrt{1-p^2}}{dp}$ datur, data inter x et y relatione. Sit valor quantitatis $-\frac{dz\sqrt{1-p^2}}{dp} = Z$ functioni curvæ z, $\int Tdz = Z$ et fumtis fluxionibus $Tdz = \overset{\prime}{Z}dz$ qua $T = \overset{\prime}{Z}$ functioni ipfius z. Si valor quantitatis $-\frac{dz\sqrt{1-p^2}}{dp} = P$ per p expreffus, erit $\int Tdz = P$ fumtifque fluxionibus $Tdz = \overset{\prime}{P}dp$ et $T = \frac{\overset{\prime}{P}dp}{dz}$, quæ functio eft quantitatis p, in poteftate femper eft $\frac{dp}{dz}$ per p exprimere.

Schol. 2. Hujus etiam theorematis fubfidio inveniri poffunt curvæ ex data relatione inter T et z, R et z, R et y, et R et p. Si enim fit $T = Z$ functioni quantitatis z, erit $\int Tdz = \int Zdz + A$, vi theorematis $\frac{dz}{\int Zdz + A} \left(= \frac{dz}{\int Tdz}\right) = -\frac{dp}{\sqrt{1-p^2}}$ et integratione $\int \frac{dz}{\int Zdz+A} + C = -\frac{dp}{\sqrt{1-p^2}}$. Pofita $\int \frac{dz}{\int Zdz+A} + C = b$ et N numerus

merus cujus logarithmus hyperbolicus 1 habetur $\sqrt{1-p^2} = \frac{N^{b\sqrt{-1}} - N^{-b\sqrt{-1}}}{2\sqrt{-1}}$ et $p = \frac{N^{b\sqrt{-1}} + N^{-b\sqrt{-1}}}{2}$, quæ functiones sunt quantitatis z, quibus positis $\overset{|}{Z}$ et $\sqrt{1-\overset{|}{Z}^2}$ respective proveniunt $x(=\int dz\sqrt{1-p^2}) = \int Z dz$ et $y(=\int p dz) = \int dz\sqrt{1-\overset{|}{Z}^2}$ quarum alterutra curvarum indoles innotescit.

Si $R = X$ functioni abscissæ x provenit $\frac{dx}{X}\left(=\frac{dx}{R}\right) = -dp$ et integratione $\overset{|}{X}\left(=C - \int \frac{dx}{R}\right) = p$ unde $\sqrt{1-p^2} = \sqrt{1-\overset{|}{X}^2}$ et $y\left(=\int \frac{p dx}{\sqrt{1-p^2}}\right) = \int \frac{\overset{|}{X} dx}{\sqrt{1-\overset{|}{X}^2}}$ æquatio curvæ indolem exprimens.

Et si $R = Y$ functioni ordinatæ y, habetur $\frac{dy}{Y}\left(=\frac{dy}{R}\right) = -\frac{p dp}{\sqrt{1-p^2}}$ et integratione $\overset{|}{Y}\left(=\int \frac{dy}{Y} + C\right) = \sqrt{1-p^2}$, unde $p = \sqrt{1-\overset{|}{Y}^2}$ et $x\left(=\int \frac{dy\sqrt{1-p^2}}{p}\right) = \int \frac{\overset{|}{Y} dy}{\sqrt{1-\overset{|}{Y}^2}}$ quæ exprimit naturam curvæ.

Hinc colligitur quod quoties $T dz$ perfecte integretur et $\int \frac{dz}{\int Z dz + A}$ obtineatur per arcus circulares dum aut $\int \overset{|}{Z} dz$ aut $\int dz\sqrt{1-\overset{|}{Z}^2}$ absolutam admittat integrationem, curvæ erunt rectificabiles, et algebraicæ, si relatio inter x et z vel inter y et z in relationem algebraicam inter x et y permutari possit.

Evidens etiam est quod si X functio est algebraica quantitatis x; vel Y quantitatis y, et non solum $\frac{dx}{X}$ vel $\frac{dy}{Y}$ sed etiam $\frac{\overset{|}{X} dx}{\sqrt{1-\overset{|}{X}^2}}$ vel $\frac{\overset{|}{Y} dy}{\sqrt{1-\overset{|}{Y}^2}}$ quantitates perfecte integrabiles, curvæ evadunt algebraicæ, alias transcendentes.

Exempl.

Exempl. 1. Invenienda sit curva ubi variatio curvaturæ $T = \frac{3 \cdot \overline{8a+27z}^{\frac{2}{3}} - 2a^{\frac{2}{3}}}{a^{\frac{1}{3}} \sqrt{\overline{8a+27z}^{\frac{2}{3}} - 4a^{\frac{2}{3}}}}$. Ut simplicior reddatur calculus ponatur $\overline{8a+27z}^{\frac{2}{3}} = u$ et $a^{\frac{2}{3}} = b$ erit $z = \frac{u^{\frac{3}{2}} - 86^{\frac{3}{2}}}{27}$, $dz = \frac{du\sqrt{u}}{18}$, $T = \frac{3u-2b}{\sqrt{b}\sqrt{u-4b}}$ et $\int T dz = \frac{u\sqrt{u}\sqrt{u-4b}}{18\sqrt{b}} + A$; sit constans hæc $A = a$, quod accidit evanescente $\int T dz\, u = b$, habetur per theorema $\frac{du\sqrt{b}}{u\sqrt{u-4b}} \left(= \frac{dz}{\int T dz}\right) = -\frac{dp}{\sqrt{1-p^2}}$ et integratione $\int \frac{du\sqrt{b}}{u\sqrt{u-4b}} + C = -\int \frac{dp}{\sqrt{1-p^2}}$, cujus æquationis termini quum sint arcus circulares quorum sinus $\sqrt{1-p^2} = \frac{\sqrt{u-4b}}{\sqrt{u}}$ et cosinus $p = \frac{2\sqrt{b}}{\sqrt{u}}$, posito arcu constanti $C = 0$, obtinetur $y\ (= \int p dz) = \int \frac{du\sqrt{b}}{9} + B) = \frac{\sqrt{u-4b}\,\sqrt{b}}{9}$ nam $B = \frac{4b\sqrt{b}}{9}$, posita $y = 0$ et $u = 4b$, atque $x (= \int dz\sqrt{1-p^2}) = \int \frac{du\sqrt{u-4b}}{18} = \frac{\overline{u-4b}^{\frac{3}{2}}}{27}$ quibus æquationibus exterminata u et substituta a habetur $y^3 = ax^2$ æquatio pro parabola cubica.

Exempl. 2. Si sit variatio curvaturæ $T = \frac{2z}{a}$ erit $\int T dz \left(= \int \frac{2z dz}{a}\right) = \frac{z^2}{a} + A$ et si $Z = 0$ $\int T dz = a$ erit constans $A = a$, atque vi theorematis $\frac{a dz}{a^2+z^2} \left(= \frac{dz}{\int T dz}\right) = -\frac{dp}{\sqrt{1-p^2}}$ et integratione $\int \frac{a dz}{a^2+z^2} + C = -\int \frac{dp}{\sqrt{1-p^2}}$; posito arcu constanti $C = 0$ cæteri sunt æquales eorumque sinus et cosinus, unde $\sqrt{1-p^2} = \frac{z}{\sqrt{a^2+z^2}}$, $p = \frac{a}{\sqrt{a^2+z^2}}$ et $dx\ (= dz\sqrt{1-p^2}) = \frac{z dz}{\sqrt{a^2+z^2}}$ et dy (=

$pdz) = \frac{adz}{\sqrt{a^2 + z^2}}$, quibus conſtat curvam eſſe catenariam.

Exempl. 3. Sit variatio curvaturæ $T = \frac{a-z}{\sqrt{2az - z^2}}$, evadit $\int Tdz = \sqrt{2az - z^2}$, per theorema $\frac{dz}{\sqrt{2az - z^2}} \left(= \frac{dz}{\int Tdz}\right) = -\frac{dp}{\sqrt{1-p^2}}$ et per integrationem $\int \frac{dz}{\sqrt{2az - z^2}} + C = -\int \frac{dp}{\sqrt{1-p^2}}$, ſi arcus ille conſtans $C = 0$, cæteri ſunt æquales eorumque ſinus et coſinus, quo $\sqrt{1-p^2} = \frac{\sqrt{2ax - x^2}}{a}$, $p = \frac{a-z}{a}$ et $y \left(= \int pdz\right) = \int \frac{a-zdz}{a}\Big) = \frac{2az - z^2}{a}$ æquatio pro cycloide ordinaria.

THEOREMA II.

Manentibus antea adhibitis denominationibus erit $\frac{dx}{y + \int Tdx} = -\frac{dp}{\sqrt{1-p^2}}$.

Quoniam $\frac{dx}{R} = -dp$, erit dividendo per $\sqrt{1-p^2}$, $\frac{dx}{R\sqrt{1-p^2}} = -\frac{dp}{\sqrt{1-p^2}}$. Propter $1 : \sqrt{1-p^2} :: CD(R) : CF = R\sqrt{1-p^2}$, ſed $dz : dx :: Tdz : Tdx$, quæ fluxio eſt ipſius DE, quare $DE = \int Tdx$, unde $CF = y + \int Tdx$ qua pro $R\sqrt{1-p^2}$ ſubſtituta, prodit $\frac{dx}{y + \int Tdx} = -\frac{dp}{\sqrt{1-p^2}}$.

Cor. 1. Quantitas $dy + Tdx$ ſemper eſt perfecte integrabilis. Nam $Tdx = -\frac{ddx\sqrt{1-p^2}}{dp}$ et $dy = \frac{pdx}{\sqrt{1-p^2}}$ unde $dy + Tdx = \frac{pdx}{\sqrt{1-p^2}} - \frac{ddx\sqrt{1-p^2}}{dp}$ et integratione $y + \int Tdx = -\frac{dx\sqrt{1-p^2}}{dp}$.

Cor.

Cor. 2. Dicatur ſemichorda curvaturæ CF F, obtinetur $\frac{dx}{F} = -\frac{dp}{\sqrt{1-p^2}}$, $\frac{dy}{F} = -\frac{pdp}{1-p^2}$ et $\frac{dz}{F} = -\frac{dp}{1-p^2}$.

Cor. 3. Si Tangens anguli BCD per r, Secans per s deſignentur habetur $\frac{dx}{y+\int Tdx} = -\frac{dr}{1+r^2}$ et $\frac{dx}{y+\int Tdx} = -\frac{ds}{s\sqrt{s^2-1}}$.

Schol. 1. Per hoc theorema via etiam patet calculandi generaliter variationem curvaturæ. Eſt enim $y+\int Tdx = -\frac{dx\sqrt{1-p^2}}{dp}$, quantitas vero $\frac{dx\sqrt{1-p^2}}{dp}$ datur data inter x et p relatione. Sit valor quantitatis $-\frac{dx\sqrt{1-p^2}}{dp} = X$ functioni abſciſſæ x æquatione ad curvam inventus, erit $\int Tdx = X - y$ et ſumtis fluxionibus $Tdx = \dot{X}dx - dy$, qua $T = \dot{X} - \frac{dy}{dx}$ ubi tam $\dot{X}$ quam $\frac{dy}{dx}$ ſunt functiones abſciſſæ x. Si valor quantitatis $-\frac{dx\sqrt{1-p^2}}{dp} = P$ per p expreſſus, erit $\int Tdx = P - y$ ſumtiſque fluxionibus $Tdx = \dot{P}dp - dy$, qua $T = \frac{\dot{P}dp}{dx} - \frac{p}{\sqrt{1-p^2}}$ ubi $\frac{\dot{P}dp}{dx}$ functio eſt quantitatis p, nam $\frac{dp}{dx}$ per p exprimi poteſt.

Schol. 2. Hoc adhibito theoremate inveniri etiam poſſunt curvæ, ex data relatione inter T et x, F et x, F et y, F et z, et F et p. Poſita enim T functione quantitatis x, patet per curvarum quadraturas, aut perfectam aut imperfectam quantitatis Tdx obtineri integrationem. Sit $\int Tdx = \dot{X} + \int \dot{X}dx$ functioni vel algebraicæ vel ex parte tranſcendenti ipſius x, cujus terminis homogeneus valor ipſius $y = \int \ddot{X}dx$ capiatur, iſque ejus indolis ut $\int \overline{\dot{X}+\ddot{X}}dx$, vel quod idem eſt $y + \int Tdx =$

3 X+

$X+\int \overline{\overset{|}{X}+\overset{||}{X}}dx$ integratione abſoluta habeatur, permanente $Tdz=Tdx\sqrt{1-\overset{||}{X}^2}$ perfecte integrabili. Per theorema deinde habetur $\frac{dx}{X+\int\overline{\overset{|}{X}+\overset{||}{X}}dx}\left(=\frac{dx}{y+\int Tdx}\right)=-\frac{dp}{\sqrt{1-p^2}}$, et per integrationem $\int\frac{dx}{X+\int\overline{\overset{|}{X}+\overset{||}{X}}dx}+C=-\int\frac{dp}{\sqrt{1-p^2}}$, ſi ponatur $\int\frac{dx}{X+\int\overline{\overset{|}{X}+\overset{||}{X}}dx}+C=k$ et N baſi logarithmorum hyperbolicorum, erit $\sqrt{1-p^2}=\frac{N^{k\sqrt{-1}}-N^{-k\sqrt{-1}}}{2\sqrt{-1}}$ et $p=\frac{N^{k\sqrt{-1}}+N^{-k\sqrt{-1}}}{2}$, $\sqrt{1-p^2}$ et p igitur ſunt functiones ipſius x, quæ ſi ponantur $\sqrt{1-\overset{|||}{X}^2}$ et $\overset{|||}{X}$, habetur $y\left(=\int\frac{pdx}{\sqrt{1-p^2}}\right)=\int\frac{\overset{|||}{X}dx}{\sqrt{1-\overset{|}{X}^2}}$, æquatio qua curvæ internoſcuntur.

Si ſit $F=Y$ functioni quantitatis y erit per Cor. 2. $\frac{dy}{Y}\left(=\frac{dy}{F}\right)=-\frac{dp}{1-p^2}$ et integratione $\int\frac{dy}{Y}+\log.C=\log.\sqrt{1-p^2}$, ponatur $\int\frac{dy}{Y}=k$ et N logarithmorum baſi, erit facto ad quantitates abſolutas tranſitu $CN^k=\sqrt{1-p^2}$, $p=\sqrt{1-C^2N^{2k}}$ et $x\left(=\int\frac{dy\sqrt{1-p^2}}{p}\right)=\int\frac{CN^kdy}{\sqrt{1-C^2N^{2k}}}$, æquatio quæ indolem curvæ indigitat.

Si $F=Z$ functioni ipſius z erit $\frac{dz}{Z}\left(=\frac{dz}{F}\right)=-\frac{dp}{1-p^2}$ et integratione $\int\frac{dz}{Z}+\log.C=\log.\sqrt{\frac{1-p}{1+p}}$, et ſi $\int\frac{dz}{Z}=k$ et N baſi logarithmica habetur $p=\frac{1-C^2N^{2k}}{1+C^2N^{2k}}$ et $y=\int pdz)=\int\frac{1-C^2N^{2k}dz}{1+C^2N^{2k}}$ qua curvæ cognoſcuntur.

Conſtat hinc quod quoties $X + \int \overset{\text{I}}{X} dx$ perfecta integratione habeatur $\int \frac{dx}{X + \int \overset{\text{I}}{X} + \overset{\text{II}}{X} dx}$ per arcus circulares dum $\frac{\overset{\text{III}}{X} dx}{\sqrt{1 - \overset{\text{III}}{X}{}^2}}$ abſolutam admittat integrationem curva ſit algebraica, ſi vero aliter evenerit tranſcendens.

Quoties $\frac{dy}{Y}$ ſit integrale logarithmicum et $\frac{CN^k dy}{\sqrt{1 - C^2 N^{2k}}}$ abſolutam admittat integrationem curva eſt algebraica, in aliis caſibus tranſcendens.

Et quoties $\int \frac{dz}{Z}$ per logarithmos inveniatur, $\frac{1 - C^2 N^{2k} dz}{1 + C^2 N^{2k}}$ abſolute ſit integrabilis pariter ac $\frac{2CN^k dz}{1 + C^2 N^{2k}}$ curva eſt algebraica, alias tranſcendens.

Exempl. 1. Si ſit variatio curvaturæ $T = \frac{3 \cdot \overline{b^2 - a^2} x \sqrt{\overline{a^2 - x^2}}}{a^3 b}$ erit $\int T dx \left(= \frac{\overline{a^2 - b^2} \cdot \overline{a^2 - x^2} \sqrt{\overline{a^2 - x^2}}}{a^3 b}\right) = \frac{a \sqrt{\overline{a^2 - x^2}}}{b} - \frac{x^2 \sqrt{\overline{a^2 - x^2}}}{ab} - \frac{b \sqrt{\overline{a^2 - x^2}}}{a} + \frac{b x^2 \sqrt{\overline{a^2 - x^2}}}{a^3}$, ſi ponatur $y = \frac{b \sqrt{\overline{a^2 - x^2}}}{a}$ habetur $y + \int T dx = \frac{a^4 + \overline{b^2 - a^2} x^2 \sqrt{\overline{a^2 - x^2}}}{a^3 b}$, adhibendo theorema $\frac{a^3 b dx}{a^4 + \overline{b^2 - a^2} x^2 \sqrt{\overline{a^2 - x^2}}} \left(= \frac{dx}{y + \int T dx}\right) = -\frac{dp}{\sqrt{1 - p^2}}$ et integrando $\int \frac{a^3 b dx}{a^4 + \overline{b^2 - a^2} x^2 \sqrt{\overline{a^2 - x^2}}} + C = -\int \frac{dp}{\sqrt{1 - p^2}}$, cujus termini ſunt arcus circulares quorum ſinus $\sqrt{1 - p^2} = \frac{a \sqrt{\overline{a^2 - x^2}}}{\sqrt{a^4 + \overline{b^2 - a^2} x^2}}$ et coſinus $p = \frac{bx}{\sqrt{a^4 + \overline{b^2 - a^2} x^2}}$ evaneſcente arcu conſtanti C, quare $y \left(= \int \frac{p dx}{\sqrt{1 - p^2}}\right) = \int \frac{b dx}{a \sqrt{\overline{a^2 - x^2}}}) = \frac{b \sqrt{\overline{a^2 - x^2}}}{a}$ et in hoc caſu curva eſt ellipſis.

Exempl. 2. Sit jam variatio curvaturæ $T=\frac{2\sqrt{2ax+x^2}}{a}$ erit $\int T dx = \frac{x\sqrt{2ax+x^2}}{a} + \int \frac{x dx}{\sqrt{2ax+x^2}}$ et poſita $y = \int \frac{a dx}{\sqrt{2ax+x^2}}$ perfecta integratione habetur $y + \int T dx = \frac{\overline{a+x}\sqrt{2ax+x^2}}{a}$. Theorematis itaque auxilio erit $\frac{a dx}{\overline{a+x}\sqrt{2ax+x^2}} \left(= \frac{dx}{y+\int T dx}\right) = -\frac{dp}{\sqrt{1-p^2}}$, et integratione $\int \frac{a dx}{\overline{a+x}\sqrt{2ax+x^2}} = C = -\int \frac{dp}{\sqrt{1-p^2}}$, ſi vero arcus ille conſtans $C=0$ cæteri ſunt æquales eorumque ſinus et coſinus, unde $\sqrt{1-p^2} = \frac{\sqrt{2ax+x^2}}{a+x}$, $p=\frac{a}{a+x}$ et $y\left(=\int \frac{p dx}{\sqrt{1-p^2}}\right) = \int \frac{a dx}{\sqrt{2ax+x^2}}$, æquatio indicans curvam eſſe catenariam.

THEOREMA III.

Dicatur coſinus anguli BCD q, poſito radio 1, cæteriſque manentibus denominationibus erit $\frac{dy}{\int T dy - x} = \frac{dq}{\sqrt{1-q^2}}$.

Eſt enim $\frac{ay}{R} = aq$, qua per $\sqrt{1-q^2}$ diviſa, dat $\frac{dy}{R\sqrt{1-q^2}} = \frac{dq}{\sqrt{1-q^2}}$; et ob $1 : \sqrt{1-q^2} :: CD\ (R) : CG = R\sqrt{1-q^2}$, ſed $dz : dy :: T dz : T dy$ cujus integrale eſt $AE = \int T dy$, unde $CG\ (= AE - AB) = \int T dy - x$, qua pro $R\sqrt{1-q^2}$ ſubſtituta, prodit $\frac{dy}{\int T dy - x} = \frac{dq}{\sqrt{1-q^2}}$.

Cor. 1. Semper $T dy - dx$ admittit perfectam integrationem. Etenim $T dy = \frac{ddy\sqrt{1-q^2}}{dq}$ et $dx = \frac{q dy}{\sqrt{1-q^2}}$, quibus $T dy - dx = \frac{ddy\sqrt{1-q^2}}{dq} - \frac{q dy}{\sqrt{1-q^2}}$ et integratione $\int T dy - x = \frac{dy\sqrt{1-q^2}}{aq}$.

Cor. 2. Dicatur semichorda curvaturæ CG G, habetur $\frac{dy}{G}=\frac{dq}{\sqrt{1-q^2}}$, $\frac{dx}{G}=\frac{qdq}{1-q^2}$ et $\frac{dz}{G}=\frac{dq}{1-q^2}$.

Cor. 3. Dicatur cotangens anguli BCD t, et cosecans v erit $\frac{dy}{\int Tdy-x}=\frac{dt}{1+t^2}$ et $\frac{dy}{\int Tdy-x}=\frac{dv}{v\sqrt{v^2-1}}$.

Schol. 1. Quoniam $\int Tdy-x=\frac{dy\sqrt{1-q^2}}{dq}$ datur ex data relatione inter y et q, sit $\frac{dy\sqrt{1-q^2}}{dq}=Y$ functioni ordinatæ y erit $\int Tdy=Y-x$ sumtisque fluxionibus $Tdy=\dot{Y}dy-dx$ qua $T=\dot{Y}-\frac{dx}{dy}$ functioni ipsius y. Si autem $\frac{dy\sqrt{1-q^2}}{dq}=Q$ functioni ipsius q erit $\int Tdy=Q-x$ et sumtis fluxionibus $Tdy=\dot{Q}dq-dx$, qua habetur $T=\frac{\dot{Q}dq}{dy}-\frac{q}{\sqrt{1-q^2}}$ per q.

Schol. 2. Hujus theorematis auxilio elicere licet curvas data relatione inter T et y, G et y, G et x, G et z, et G et q. Si enim sit T functio ipsius y generaliter $\int Tdy=Y+\int\dot{Y}dy+A$, quæ functio est algebraica ipsius y quoties $\int\dot{Y}dy$ absolute sumi possit. Assumatur $x=\int\ddot{Y}dy$, tali ipsius y functioni ut non solum $\int Tdy-x=Y+\int\overline{\dot{Y}+\ddot{Y}}dy$ sed etiam $\int Tdz=\int Tdy\sqrt{1-\ddot{Y}^2}$ absoluta integratione habeantur, provenit vi theorematis $\frac{dy}{Y+\int\overline{\dot{Y}+\ddot{Y}}dy+A}\left(=\frac{dy}{\int Tdy-x}\right)=\frac{dq}{\sqrt{1-q^2}}$ et integratione $\int\frac{dy}{Y+\int\overline{\dot{Y}+\ddot{Y}}dy+A}+C=\int\frac{dq}{\sqrt{1-q^2}}$. Posita $\frac{dy}{Y+\int\overline{\dot{Y}+\ddot{Y}}dy+A}+C=l$ et N basi logarithmica erit $q=\frac{N^{l\sqrt{-1}}-N^{-l\sqrt{-1}}}{2\sqrt{-1}}$ et $\sqrt{1-q^2}$

$= \frac{N^{l\sqrt{-1}} + N^{-l\sqrt{-1}}}{2}$ quæ functiones sunt quantitatis y, quibus positis $\overset{\text{III}}{Y}$ et $\sqrt{1 - \overset{\text{III}}{Y}{}^2}$ prodit $x \left(= \int \frac{q\,dy}{\sqrt{1-q^2}}\right) = \int \frac{\overset{\text{III}}{Y}dy}{\sqrt{1-\overset{\text{III}}{Y}{}^2}}$ æquatio quæ indolem curvarum indicat.

Si G=X functioni ipsius x erit per Cor. 2. $\frac{dx}{X} \left(= \frac{dx}{G}\right) = \frac{q\,dq}{1-q^2}$, et integratione log. $CN^l \left(= \int \frac{dx}{X} + \log. C\right) = \log. \frac{1}{\sqrt{1-q^2}}$ si $\int \frac{dx}{X} = l$, exinde $\sqrt{1-q^2} = \frac{1}{CN^l}$, $q = \frac{\sqrt{C^2N^{2l}-1}}{CN^l}$ et $y \left(= \int \frac{dx\sqrt{1-q^2}}{q}\right) = \int \frac{dx}{\sqrt{C^2N^{2l}-1}}$, quæ curvæ naturam indigitat.

Si G=Z functioni ipsius z erit $\frac{dz}{Z} \left(= \frac{dz}{G}\right) = \frac{dq}{1-q^2}$, et integratione log. $CN^l \left(= \frac{dz}{Z} + C\right) = \log. \sqrt{\frac{1+q}{1-q}}$ si $\int \frac{dz}{Z} = l$, unde $q = \frac{C^2N^{2l}}{1+C^2N^{2l}} \sqrt{1-q^2} = \frac{2CN^l}{1+C^2N^{2l}}$ $x \left(= \int q\,dz\right) = \int \frac{C^2N^{2l}-1\,dz}{1+C^2N^{2l}}$ et $y \left(= \int dz\sqrt{-1q^2}\right) = \int \frac{2CN^l dz}{1+C^2N^{2l}}$ quibus curvæ cognoscuntur.

Patet hinc quod quando $Y + \int \overset{\text{I}}{Y} dy$ algebraice habeatur $\int \frac{dy}{Y + \int \overset{\text{I}}{Y} + \overset{\text{II}}{Y} dy + A}$ per quadraturam circuli, et $\int \frac{\overset{\text{III}}{Y}dy}{\sqrt{1-\overset{\text{III}}{Y}{}^2}}$ etiam obtineatur algebraice, curvæ evadunt algebraicæ, secus vero transcendentes.

Quando $\int \frac{dx}{X}$ vel $\int \frac{dz}{Z}$ obtineatur per logarithmos, et $\int \frac{dx}{\sqrt{C^2N^{2l}-1}}$, vel tam $\int \frac{C^2N^{2l}-1\,dz}{1+C^2N^{2l}}$ quam $\int \frac{2CN^l dz}{1+C^2N^{2l}}$ absoluta integratione, curvæ erunt algebraicæ.

Exempl.

Exempl. 1. Sit index variationis curvaturæ $T = \frac{6y}{a}$ erit $\int T dy = \frac{3y^2}{a} + A$, ſi quantitas illa conſtans $A = \frac{a}{2}$ quod evenit quum $\int T dy = \frac{a}{2}$ et $y = 0$; ſumatur $x = \frac{y^2}{a}$ erit vi theorematis $\frac{2ady}{a^2+4y^2}$ $\left(= \frac{dy}{\int T dy - x}\right) = \frac{dq}{\sqrt{1-q^2}}$ et integratione $\int \frac{2ady}{a^2+4y^2} + C = \int \frac{dq}{\sqrt{1-q^2}}$, cujus æquationis termini quoniam ſint arcus circulares quorum ſinus $q = \frac{2y}{\sqrt{a^2+4y^2}}$ et coſinus $\sqrt{1-q^2} = \frac{a}{\sqrt{a^2+4y^2}}$, arcu conſtanti $C = 0$, obtinetur $x \left(= \int \frac{qdy}{\sqrt{1-q^2}}\right) = \frac{y^2}{a}$ æquatio pro parabola Apolloniana.

Exempl. 2. Si ſit $T = \frac{a^2}{y\sqrt{a^2-y^2}}$ habetur $\int T dy = \int \frac{dy\sqrt{a^2-y^2}}{y} - \sqrt{a^2-y^2} + A$, ſi quantitas illa conſtans $A = 0$ quod evenit quum $\int T dy = 0$ et $y = a$, et aſſumatur $x = \int \frac{dy\sqrt{a^2-y^2}}{y}$, evadit per theorema $-\frac{dy}{\sqrt{a^2-y^2}} \left(= \frac{dy}{\int T dy - x}\right) = \frac{dq}{\sqrt{1-q^2}}$, et per integrationem $-\int \frac{dy}{\sqrt{a^2-y^2}} + C = \int \frac{dq}{\sqrt{1-q^2}}$, quorum arcuum ſinus $q = \frac{\sqrt{a^2-y^2}}{a}$ et coſinus $\sqrt{1-q^2} = \frac{y}{a}$ ſi conſtans ille $C = 0$, atque inde $dx \left(\frac{qdy}{\sqrt{1-q^2}}\right) = \frac{dy\sqrt{1-y^2}}{y}$ qua patet curvam eſſe tractoriam.

THEOREMA IV.

Dicatur ſumma tangentium angulorum HCD et BCD H, et differentia tangentium angulorum HCD et CKB K, retentis reliquis denominationibus erit $\frac{dx}{\int H dx} = -\frac{dp}{\sqrt{1-p^2}}$ et $\frac{dy}{\int K dy} = \frac{dq}{\sqrt{1-q^2}}$.

Quoniam

Quoniam $dy=\frac{pdx}{\sqrt{1-p^2}}$ erit $dy+Tdx=\overline{T+\frac{p}{\sqrt{1-p^2}}}\,dx$ et quum $H=T+\frac{p}{\sqrt{1-p^2}}$ habetur $dy+Tdx=Hdx$. Eodem modo quum $dx=\frac{qdy}{\sqrt{1-q^2}}$ erit $\int Tdy-dx=T-\frac{q}{\sqrt{1-q^2}}dy$, ſed $K=T-\frac{q}{\sqrt{1-q^2}}$, unde $\int Tdy-x=Kdy$. Per theorema igitur 2 et 3 provenit $\frac{dx}{\int Hdx}=-\frac{dp}{\sqrt{1-p^2}}$ et $\frac{dy}{\int Kdy}=\frac{dq}{\sqrt{1-q^2}}$.

Cor. Si ſit ut antea tangens anguli BCD r, cotangens t, ſecans s, et coſecans v, erit $\frac{dx}{\int Hdx}=-\frac{dr}{r+r^2}$ et $\frac{dx}{\int Hdx}=-\frac{ds}{s\sqrt{s^2-1}}$, $\frac{dy}{\int Kdy}=\frac{dt}{1+t^2}$ et $\frac{dy}{\int Kdy}=\frac{dv}{v\sqrt{v^2-1}}$.

Schol. Ope hujus theorematis invenire licet curvas, data relatione inter H et x atque K et y. Itaque ſit $H=X$ functioni ipſius x erit $\int Hdx=\int Xdx+A$, vi theorematis $\frac{dx}{\int Xdx+A}$ $\left(=\frac{dx}{\int Hdx}\right)=-\frac{dp}{\sqrt{1-p^2}}$, et integratione $\int\frac{dx}{\int Xdx+A}+C=-\int\frac{dp}{\sqrt{1-p^2}}$. Poſita $\int\frac{dx}{\int Xdx+A}+C=m$, et N logarithmorum baſi prodit $\sqrt{1-p^2}=\frac{N^{m\sqrt{-1}}-N^{-m\sqrt{-1}}}{2\sqrt{-1}}$ et $p=\frac{N^{m\sqrt{-1}}+N^{-m\sqrt{-1}}}{2}$, quibus functionibus quantitatis x poſitis $\sqrt{1-\overset{1}{X}{}^2}$ et $\overset{1}{X}$ provenit æquatio $y\left(=\int\frac{pdx}{\sqrt{1-p^2}}\right)=\frac{\overset{1}{X}dx}{\sqrt{1-\overset{1}{X}{}^2}}$ naturam curvarum exprimens.

Si $K=Y$ functioni quantitatis y, eadem calculandi ratione habetur $x\left(=\int\frac{qdy}{\sqrt{1-q^2}}\right)=\frac{\overset{1}{Y}dy}{\sqrt{1-\overset{1}{Y}{}^2}}$ æquatio qua curvæ cognoſcuntur.

Quando

Quando $\int Xdx$ vel $\int Ydy$ absoluta integratione, $\int \frac{dx}{\int Xdx + A}$ vel $\int \frac{dy}{\int Ydy + A}$ per rectificationem circuli, et $\int \frac{\dot{X}dx}{\sqrt{1-\dot{X}^2}}$ vel $\int \frac{\dot{Y}dy}{\sqrt{1-\dot{Y}^2}}$ integratione perfecta obtineantur, curva est algebraica.

Exempl. 1. Si sit $H = \frac{a+12x}{2\sqrt{a}\sqrt{x}}$ erit $\int Hdx = \frac{\overline{a+4x}\sqrt{x}}{\sqrt{a}} + A$, et posita $A = 0$ habetur per theorema $\frac{dx\sqrt{a}}{a+4x\sqrt{x}} \left(= \frac{dx}{\int Hdx}\right) = -\frac{dp}{\sqrt{1-p^2}}$ et per integrationem $\int \frac{dx\sqrt{a}}{a+4x\sqrt{x}} + C = \int \frac{dp}{\sqrt{1-p^2}}$, cujus termini quum sint arcus circulares quorum sinus $\sqrt{1-p^2} = \frac{2\sqrt{x}}{\sqrt{a+4x}}$ et cosinus $p = \frac{\sqrt{a}}{\sqrt{a+4x}}$, posita $C = 0$, obtinetur $y \left(= \int \frac{pdx}{\sqrt{1-p^2}}\right) = \sqrt{ax}$, quæ parabolam Apolloniam exprimit.

Exempl. 2. Sit $H = \frac{2a^4 - x^4}{ax^2\sqrt{a^2-x^2}}$ erit $\int Hdx = \frac{\overline{x^2-2a^2}\sqrt{a^2-x^2}}{ax} + A$, et si $A = 0$, per theorema $\frac{axdx}{x^2+2a^2\sqrt{a^2-x^2}} \left(= \frac{dx}{\int Hdx}\right) = -\frac{dp}{\sqrt{1-p^2}}$ et per integrationem $\int \frac{axdx}{x^2-2a^2\sqrt{a^2-x^2}} + C = -\int \frac{dp}{\sqrt{1-p^2}}$, et si $C = 0$, $\sqrt{1-p^2} = \frac{\sqrt{a^2-x^2}}{\sqrt{2a^2-x^2}}$ $p = \frac{a}{\sqrt{2a^2-x^2}}$ et $y \left(= \int \frac{pdx}{\sqrt{1-p^2}}\right) = \int \frac{adx}{\sqrt{a^2-x^2}}$ æquatio pro curva sinuum.

Exempl. 3. Si sit $K = \frac{\overline{5a^2+6y^2}.y}{a\sqrt{a^2+y^2}}$ erit $\int Kdy = \frac{\overline{a^2+2y^2}\sqrt{a^2+y^2}}{a^2} + A$, si $A = 0$ habetur per theorema $\frac{a^2dy}{a^2+2y^2\sqrt{a^2+y^2}} \left(= \frac{dy}{\int Kdy}\right) = \frac{dq}{\sqrt{1-q^2}}$ et integratione $\int \frac{a^2dy}{a^2+2y^2\sqrt{a^2+y^2}} + C = -\int \frac{dp}{\sqrt{1-p^2}}$, qua $q = \frac{y}{\sqrt{a^2+2y^2}}$, $\sqrt{1-q^2} = \frac{\sqrt{a^2+y^2}}{\sqrt{a^2+2y^2}}$, si $C = 0$, unde $x \left(= \int \frac{qdy}{\sqrt{1-q^2}}\right) = \sqrt{a^2+y^2}$ æquatio pro hyperbola æquilatera.

Exempl.

Exempl. 4. Sit $K = \frac{y}{\sqrt{a^2 - y^2}}$ erit $\int K dy = A - \sqrt{a^2 - y^2}$ et si $A = 0$, per theorema $-\frac{dy}{\sqrt{a^2 - y^2}} \left(= \frac{dy}{\int K dy}\right) = \frac{dq}{\sqrt{1 - q^2}}$ et per integrationem $-\int \frac{dy}{\sqrt{a^2 - y^2}} + C = \int \frac{dq}{\sqrt{1 - q^2}}$ qua $q = \frac{\sqrt{a^2 - y^2}}{a}$, $\sqrt{1 - q^2} = \frac{y}{a}$ et $dx \left(= \frac{q dy}{\sqrt{1 - q^2}}\right) = \frac{dy \sqrt{a^2 - y^2}}{y}$ quæ Tractoriam exprimit.

THEOREMA V.

Designetur productum tangentium angulorum HCD et BCD per U, et angulorum HCD et CKB per V cæteris manentibus erit $\frac{dx}{\int U dx - x} = -\frac{dp}{p}$ et $\frac{dy}{Y + \int V dy} = \frac{dq}{q}$.

Quoniam $dy = \frac{p dx}{\sqrt{1 - p^2}}$ et $U = \frac{Tp}{\sqrt{1 - p^2}}$ erit $T dy$ $\left(= \frac{Tp dx}{\sqrt{1 - p^2}}\right) = U dx$, et integratione $\int T dy = \int U dx$ qua $\int T dy - x = \int U dx - x$. Et quoniam $dx = \frac{q dy}{\sqrt{1 - q^2}}$ et $V = \frac{Tq}{\sqrt{1 - q^2}}$ erit $T dx$ $\left(= \frac{Tq dy}{\sqrt{1 - q^2}}\right) = V dy$, $\int T dx = \int V dy$ et $y + \int T dx = y + \int V dy$. Theoremate 2. et 3. prodit $\frac{dx}{\int U dx - x} = -\frac{dp}{p}$ et $\frac{dy}{y + \int V dy} = \frac{dq}{q}$.

Cor. Si anguli BCD tangens, cotangens, &c. designentur ut antea, habetur $\frac{dx}{\int U dx - x} = -\frac{dr}{r \cdot 1 + r^2}$, $\frac{dy}{y + \int V dy} = -\frac{dt}{t \cdot 1 + t^2}$, &c.

Schol. Per hoc theorema curvæ inveniuntur ex data relatione inter U et x, atque inter V et y. Si enim sit $U = X$ functioni ipsius x erit $\int U dx = \int X dx + A$, per theorema $\frac{dx}{\int X dx - x + A}$ $\left(= \frac{dx}{\int U dx - x}\right) = -\frac{dp}{p}$, et per integrationem $\int \frac{dx}{\int X dx - x + A} + \log. C =$ log.

log. $\frac{1}{p}$. Ponatur $\int \frac{dx}{\int Xdx - x + A} = n$ et N basi logarithmica, erit $\frac{1}{p} = CN^n$, $p = \frac{1}{CN^n}$, $\sqrt{1-p^2} = \frac{\sqrt{C^2N^{2n}-1}}{CN^n}$ et y $\left(= \frac{pdx}{\sqrt{1-p^2}}\right) =$ $\int \frac{dx}{\sqrt{C^2N^{2n}-1}}$ qua æquatione curvarum indoles innotescit.

Si V = Y functioni ipsius y, eadem calculandi ratione provenit x $\left(= \int \frac{qdy}{\sqrt{1-q^2}}\right) = \int \frac{CN^n dy}{\sqrt{1-C^2N^{2n}}}$ qua curvæ cognoscuntur.

Evidens hinc est quod quoties $\int Xdx$ vel $\int Ydy$ algebraice $\int \frac{dx}{\int Xdx - x + A}$ vel $\int \frac{dy}{y + \int Ydy + A}$ per logarithmos, atque $\int \frac{dx}{\sqrt{C^2N^{2n}-1}}$ vel $\int \frac{CN^n dy}{\sqrt{1-C^2N^{2n}}}$ integratione absoluta, obtineantur, curva est algebraica.

Exempl. 1. Si sit $U = 3$ erit $\int Udx = 3x + A$, si vero $\int Udx = \frac{a}{2}$ quando $x = 0$ erit $A = \frac{a}{2}$ et $\int Udx - x = \frac{a+4x}{2}$. Per theorema igitur $\frac{2dx}{a+4x}$ $\left(= \frac{dx}{\int Udx - x}\right) = -\frac{dp}{p}$ et per integrationem log. $\sqrt{a+4x}$ + log. C = log. $\frac{1}{p}$, posita $p = 1$ dum $x = 0$ log. C = − log. $\sqrt{a}$, unde facto a logarithmis transitu $\frac{\sqrt{a+4x}}{\sqrt{a}} = \frac{1}{p}$, qua $p = \frac{\sqrt{a}}{\sqrt{a+4x}}$, $\sqrt{1-p^2} = \frac{2\sqrt{x}}{\sqrt{a+4x}}$ et y $\left(= \int \frac{pdx}{\sqrt{1-p^2}}\right) = \int \frac{dx\sqrt{a}}{2\sqrt{x}} = \sqrt{ax}$ æquatio pro Parabola Apolloniana.

Exempl. 2. Sit $U = \frac{x^3 - 4a^3}{x\sqrt{x}}$ erit $\int Udx = \frac{x^3 - 2a^3}{3x^2} + A$, si autem $\int Udx = 0$ et $x = a\sqrt[3]{2}$, erit $A = 0$ et $\int Udx - x = \frac{2 \cdot a^3 - x^3}{3x^2}$. Vi igitur theorematis erit $\frac{3x^2dx}{2a^3 - x^3}$ $\left(= \frac{dx}{\int Udx - x}\right) = -\frac{dp}{p}$, et integratione

4 log.

log. $\frac{a\sqrt{a}}{\sqrt{a^3-x^3}}$ + log. C = log. $\frac{1}{p}$ qua $p = \frac{\sqrt{a^3-x^3}}{a\sqrt{a}}$; $\sqrt{1-p^2} = \frac{x\sqrt{x}}{a\sqrt{a}}$ et $y \left(= \int \frac{pdx}{\sqrt{1-p^2}}\right) = \frac{dx\sqrt{a^3-x^3}}{x\sqrt{x}}$ æquatio ad curvam quæſitam.

Exempl. 3. Si $V = -\frac{1}{2}$ erit $\int Vdy = A - \frac{y}{2}$, poſita $\int Vdy = 0$ et $y = 0$ erit $A = 0$ et $y + \int Vdy = \frac{y}{2}$. Per theorema obtinetur $\frac{2dy}{y} \left(= \frac{dy}{y + \int Vdy}\right) = \frac{dq}{q}$ et per integrationem log. y^2 + log. C = log. q, ſi $q = 1$ et $y = a$ erit log. C = − log. a^2, unde $q = \frac{y^2}{a^2}$, $\sqrt{1-q^2} = \frac{\sqrt{a^4-y^4}}{a^2}$ atque $dx \left(= \frac{qdy}{\sqrt{1-q^2}}\right) = \frac{y^2dy}{\sqrt{a^4-y^4}}$, curva ergo eſt Elaſtica,

Exempl. 4. Sit $V = \frac{a^2-2y^2}{y^2}$ erit $\int Vdy = A - \frac{a^2+2y^2}{y}$, ſi $\int Vdy = -3a$ et $y = a$ erit $A = 0$, indeque $y + \int Vdy = -\frac{a^2+y^2}{y}$. Theorematis ope habetur $-\frac{ydy}{a^2+y^2} \left(= \frac{dy}{y + \int Vdy}\right) = \frac{dq}{q}$ et integratione log. $\frac{1}{\sqrt{a^2+y^2}}$ + log. C = log. q, ſi $q = 1$ et $y = 0$ erit log. C = log. a et exinde $q = \frac{a}{\sqrt{a^2+y^2}}$, $\sqrt{1-q^2} = \frac{y}{\sqrt{a^2+y^2}}$ et $dx \left(= \frac{qdy}{\sqrt{1-q^2}}\right) = \frac{ady}{y}$ æquatio pro Logarithmica.

THEOREMA VI.

Dicatur ED L, et AE M, retentis præterea adhibitis denominationibus erit $\frac{dL}{T} = dx$ et $\frac{dM}{T} = dy$.

Quoniam $dz : dx :: Tdz\ (dR) : Tdx$ habetur $dL = Tdx$ et

$\frac{dL}{T} = dx$. Et quoniam $dz : dy :: Tdz\ (dR) : Tdy$ obtinetur $dM = Tdy$ et $\frac{dM}{T} = dy$.

Cor. Quum $Tdy = Udx$ et $Tdx = Vdy$, erit fubftitutione $\frac{dM}{U} = dx$ et $\frac{dL}{V} = dy$.

Schol. Hoc adhibito theoremate inveniri poffunt curvæ data relatione inter T et L, T et M, atque inter U et M et V et L. Ponatur $L = \overset{\shortmid}{T}$ functioni quantitatis T habetur per theorema $\frac{d\overset{\shortmid}{T}}{T} \left(= \frac{dL}{T}\right) = dx$ et integratione $\int \frac{d\overset{\shortmid}{T}}{T} + C = x$ qua T per x datur. Curvæ deinde per theorema 2. elici poffunt.

Si $M = \overset{\shortmid}{T}$ ipfius T functioni, habetur eodem modo T per y. Si $M = \overset{\shortmid}{U}$ functioni ipfius U, obtinetur U per x, et fi $L = \overset{\shortmid}{V}$ functioni quantitatis V, datur V per y. Per theorema deinde 3. et 5. curvæ inveniuntur.

Evidens quidem eft quod curvæ effe non poffunt algebraicæ nifi $\int \frac{dL}{T}$, $\int \frac{dM}{T}$, $\int \frac{dM}{U}$ vel $\int \frac{dL}{V}$, obtineantur integratione abfoluta.

Exempl. 1. Si fit $L = \frac{aT^3}{54}$ erit $dL = \frac{aT^2 dT}{18}$, et per hoc theorema $\frac{aTdT}{18} \left(= \frac{dL}{T}\right) = dx$ et integratione $\frac{aT^2}{36} + C = x$ qua $T = \frac{6\sqrt{x}}{\sqrt{a}}$, fi $C = 0$. Per theorema 2. reperitur $y = \sqrt{ax}$, æquatio pro Parabola Apolloniana.

Exempl. 2. Si fit $M = -\int \frac{aT^2 dT}{2 \cdot \overline{1+T^2}|^2}$ erit $dM = -\frac{aT^2 dT}{2 \cdot \overline{1+T^2}|^2}$ et ope theorematis $-\frac{aTdT}{2 \cdot \overline{1+T^2}|^2} \left(= \frac{dM}{T}\right) = dy$, et integratione $\frac{a}{4 \cdot \overline{1+T^2}} + C = y$, qua fi $C = 0$, $T = \frac{\sqrt{a-4y}}{2\sqrt{y}}$. Per theorema 3. habetur

$dx =$

$dx = \frac{2dy\sqrt{y}}{\sqrt{a-4y}}$, æquatio pro Cycloide ordinaria.

Exempl. 3. Sit $L = -a\sqrt{V}$ erit $dL = -\frac{adV}{2\sqrt{V}}$ et per theorema $-\frac{adV}{2V\sqrt{V}} \left(= \frac{dL}{V}\right) = dy$ et integratione $\frac{a}{\sqrt{V}} + C = y$, et si $C = 0$, habetur $V = \frac{a^2}{y^2}$ et deinde per theorema 5. $dx = \frac{dy\sqrt{a^2-y^2}}{y}$, qua constat curvam esse Tractoriam.

THEOREMA VII.

Dicatur ut antea CF F et CG G, et summa tangentium angulorum HCD et BCD, H, et differentia tangentium angulorum HCD et CKB, K, erit $\frac{dF}{H} = dx$ et $\frac{dG}{K} = dy$.

Quoniam $dF\ (= dy + Tdx) = Hdx$ et $dG\ (= \int Tdy - x) = Kdy$ provenit $\frac{dF}{H} = dx$ et $\frac{dG}{K} = dy$.

Cor. Quum $F = -\frac{dx\sqrt{1-p^2}}{dp}$ et $G = \frac{dy\sqrt{1-q^2}}{dp}$ provenit divisione $\frac{dF}{FH} = -\frac{dp}{\sqrt{1-p^2}}$ atque $\frac{dG}{GK} = \frac{dq}{\sqrt{1-q^2}}$.

Schol. Auxilio hujus theorematis inveniuntur curvæ ex data relatione inter F et H, G et K, H et p atque K et q. Nam si sit $F = \overset{'}{H}$ functioni ipsius H, vel $G = \overset{'}{K}$ functioni ipsius K, habetur per theorema $\frac{d\overset{'}{H}}{H} \left(= \frac{dF}{H}\right) = dx$ et integratione $\int \frac{d\overset{'}{H}}{H} + C = x$ qua H per x datur. Eodem modo $\frac{d\overset{'}{K}}{K} \left(= \frac{dG}{K}\right) = dy$ et integratione $\int \frac{d\overset{'}{K}}{K} + C = y$ qua K per y obtinetur. Theorema 4. ulterius progredienti viam monstrat ad curvas inveniendas.

Patet

Patet quod curva non ſit algebraica niſi $\int \frac{dH}{H}$ vel $\int \frac{dK}{K}$ obtineantur perfecta integratione.

Exempl. 1. Si ſit $F = \frac{a}{\sqrt{1+H^2}}$ habetur per theorema – $\frac{adH}{\overline{1-H^2}|^{\frac{3}{2}}} \left(= \frac{dF}{H}\right) = dx$, et integratione $\frac{aH}{\sqrt{1-H^2}} + C = -x$ qua $H = -\frac{x}{\sqrt{a^2 - x^2}}$, poſita $C = 0$. Per theorema deinde 4. provenit $y = \sqrt{a^2 - x^2}$ æquatio pro circulo.

Exempl. 2. Sit $F = \frac{a \,.\, \overline{H^3 + H^2 + 6\sqrt{H^2 - 12}}}{108}$, erit per theorema $\frac{a \,.\, \overline{H^2 - 6 + H\sqrt{H^2 - 12}} \,.\, dH}{36\sqrt{H^2 - 12}} \left(= \frac{dF}{H}\right) = dx$ et integratione facta $\frac{a \,.\, \overline{H^2 - 6 + H\sqrt{H^2 - 12}}}{72} + C = x$, et poſita $C = 0$ habetur $H = \frac{a + 12x}{2\sqrt{a}\sqrt{x}}$, unde per theorema 4. prodit $y = \sqrt{ax}$ æquatio pro Parabola Apolloniana.

Exempl. 3. Sit $G = -\frac{a \,.\, \overline{4 + K^2}}{4}$ erit per theorema $\frac{adK}{2} \left(= \frac{dG}{K}\right) = dy$, et integratione $\frac{aK}{2} + C = y$, et ſi $C = 0$ $K = \frac{2y}{a}$ unde per theorema 4. $dx = \frac{ady}{y}$, qua conſtat curvam eſſe Logarithmicam.

THEOREMA VIII.

Dicatur ut antea productum tangentium angulorum HCD et BCD U, et productum tangentium angulorum HCD et CKB V manentibus reliquis denominationibus erit $\frac{dG}{U-1} = dx$ et $\frac{dF}{1+V} = dy$.

Quoniam

Quoniam $G = \int T dy - x$ erit $dG = T dy - dx$, ſed $T dy = U dx$, unde $dG = \overline{U - 1}\, dx$ et $\frac{dG}{U - 1} = dx$. Eeodem modo quum $F = y + \int T dx$ erit $dF = dy + T dx$, ſed $T dx = V dy$ quare $dF = \overline{1 + V}\, dy$ et $\frac{dF}{1 + V} = dy$.

Cor. Quoniam $G = \frac{dy\sqrt{1 - q^2}}{dq}$ et $F = -\frac{dx\sqrt{1 - p^2}}{dp}$, habetur ſubſtitutione debita $\frac{dG}{G \cdot \overline{U - 1}} = -\frac{dp}{p}$ et $\frac{dF}{F^2\, \overline{1 + V}} = \frac{dq}{q}$.

Schol. Ope hujus theorematis indagantur curvæ data relatione inter G et U vel inter F et V. Nam ſi ſit $G = \acute{U}$ functioni quantitatis U vel $F = \acute{V}$ functioni quantitatis V obtinetur per theorema in caſu priori $\frac{d\acute{U}}{U - 1} \left(= \frac{dG}{U - 1}\right) = dx$ et integratione $\int \frac{d\acute{U}}{U - 1} + C = x$, qua U per x habetur; in poſteriori $\frac{d\acute{V}}{1 + V} \left(= \frac{dF}{1 + V}\right) = dy$ et integratione $\int \frac{d\acute{V}}{1 + V} + C = y$, qua V habetur per y. Per theorema deinde 5. curvæ cognoſcuntur.

Datur etiam per Cor. U in p, et V in q, et conſequenter T in p vel q, nam $U = \frac{Tp}{\sqrt{1 - p^2}}$ et $V = \frac{Tq}{\sqrt{1 - q^2}}$.

Conſtat hinc quod curvæ non ſint algebraicæ niſi $\int \frac{d\acute{U}}{U - 1}$ vel $\int \frac{d\acute{V}}{1 + V}$ obtineantur integratione abſoluta.

Exempl. 1. Si ſit $G = \frac{a \cdot \overline{U - 3}}{2}$ erit per theorema $\frac{a dU}{2\, \overline{U - 1}} \left(= \frac{dG}{U - 1}\right) = dx$ et integratione log. $\overline{1 - U}$ + log. $C = \frac{2x}{a}$ et ſi $C = \frac{a^2}{2}$

log.

log. $\frac{a^2 \cdot \overline{1-U}}{2} = \frac{2x}{a}$ et $\frac{a \cdot \overline{1-U}}{2} = N^{\frac{2x}{a}}$ qua $U = \frac{a^2 - 2N^{\frac{2x}{a}}}{a^2}$. Per theorema deinde 5. habetur $dy = \frac{dxN^{\frac{x}{a}}}{a}$ qua conſtat curvam eſt Logarithmicam.

Exempl. 2. Si ſit $T = \frac{a \cdot \overline{V-1}\sqrt{V+2}}{3\sqrt{3}}$ erit per theorema $\frac{adV}{2\sqrt{3}\sqrt{V+2}} \left(= \frac{dF}{1+V}\right) = dy$ et per integrationem $\frac{a\sqrt{V+2}}{\sqrt{3}} = y$ qua $V = \frac{3y^2 - 2a^2}{a^2}$; et per theorema 5. $dx = \frac{dy\sqrt{y^2-a^2}}{a}$, æquatio ad curvam cujus conſtructio a quadratura hyperbolæ dependet.

THEOREMA IX.

Sint LC et lc duæ curvæ eandem habentes Evolutam QD, dicatur radiorum oſculi CD cD conſtans differentia cC b, curvæ lc variatio curvaturæ S, ceteriſque ut antea manentibus erit $\frac{dR}{R-bS} = -\frac{dp}{\sqrt{1-p^2}}$.

Quoniam radius curvaturæ DH evolutæ ſit $RT = R - bS$, erit $\frac{1}{R-bS} = \frac{1}{RT}$, quæ per $dR\ (=Tdz) = -\frac{RTdp}{\sqrt{1-p^2}}$ multiplicata, moſtrat eſſe $\frac{dR}{R-bS} = -\frac{dp}{\sqrt{1-p^2}}$.

Cor. Si ſint ut antea tangens anguli BCD r et ſecans s, habetur $\frac{dR}{R-bS} = -\frac{dr}{1+r^2}$ et $\frac{dR}{R-bS} = -\frac{ds}{S\sqrt{S^2-1}}$.

Schol. Subſidio hujus theorematis invenire licet curvas, data relatione inter S et R vel inter S et T nam $\frac{S}{T} = \frac{R}{R-b}$. Itaque ſi

ponatur

ponatur $S=\overset{|}{R}$ functioni radii curvedinis R, erit $\frac{dR}{R-b\overset{|}{R}}\left(=\frac{dR}{R-bS}\right)$ $=-\frac{dp}{\sqrt{1-p^2}}$, et integratione $\int\frac{dR}{R-b\overset{|}{R}}+C=-\int\frac{dp}{\sqrt{1-p^2}}$. Sit $\int\frac{dR}{R-b\overset{|}{R}}+C=f$ et N logarithmorum basi habetur $\sqrt{1-p^2}=$ $\frac{N^{f\sqrt{-1}}-N^{-f\sqrt{-1}}}{2\sqrt{-1}}$ et $p=\frac{N^{f\sqrt{-1}}+N^{-f\sqrt{-1}}}{2}$ functionibus quantitatis R, quibus R per p exprimi potest. Per theorema igitur 1. curvas internoscere valemus.

Si $R=\overset{|}{S}$ functioni quantitatis S habetur $\frac{d\overset{|}{S}}{\overset{|}{S}-bS}\left(=\frac{dR}{R-bS}\right)$ $=-\frac{dp}{\sqrt{1-p^2}}$, et integratione $\int\frac{d\overset{|}{S}}{\overset{|}{S}-bS}+C=-\int\frac{dp}{\sqrt{1-p^2}}$, posita $\int\frac{d\overset{|}{S}}{\overset{|}{S}-bS}+C=g$, erit $\sqrt{1-p^2}=\frac{N^{g\sqrt{-1}}-N^{-g\sqrt{-1}}}{2-\sqrt{1}}$ et $p=\frac{N^{g\sqrt{-1}}+N^{-g\sqrt{-1}}}{2}$ quibus S per p datur. Per theoremata Partis I. invenire licet curvas omnes eandem evolutam habentes.

Hinc videtur, quod curvæ non sint algebraicæ nisi $\int\frac{dR}{R-b\overset{|}{R}}$ vel $\int\frac{d\overset{|}{S}}{\overset{|}{S}-bS}$ per circuli rectificationem obtineatur.

Exempl. 1. Si sit $S=\frac{2R}{\sqrt{a}\,.\sqrt{R-a}}$ supposita $b=a$, erit per theorema $\frac{dR\sqrt{a}}{2R\sqrt{R-a}}\left(=\frac{dR}{R-bS}\right)=-\frac{dp}{\sqrt{1-p^2}}$ et integratione $\int\frac{dR\sqrt{a}}{2R\sqrt{R-a}}+C=-\int\frac{dp}{\sqrt{1-p^2}}$, si vero arcus ille constans $C=0$ erit $\sqrt{1-p^2}=\frac{\sqrt{R-a}}{\sqrt{R}}$ qua $R=ap^2$, et per Cor. 1. Theor. 1. habetur $dy=\frac{adx}{\sqrt{x^2-a^2}}$, æquatio pro Catenaria.

 Exempl.

Exempl. 2. Sit $S=\frac{5a^2+R^2}{a\cdot\overline{a-5R}}$, poſita $b=\frac{a}{5}$ erit per theorema $-\frac{adR}{a^2+R^2}\left(=\frac{dR}{R-bS}\right)=-\frac{dp}{\sqrt{1-p^2}}$ et facta integratione $-\int\frac{adR}{a^2+R^2}+C$ $=-\int\frac{dp}{\sqrt{1-p^2}}$, qua ſi $C=0$, habetur $\sqrt{1-p^2}=\frac{R}{\sqrt{a^2+R^2}}$ et $R=\frac{a\sqrt{1-p^2}}{p}$. Per theorema 1. $dx=\frac{dy\sqrt{a^2-y^2}}{y}$ qua conſtat curvam eſſe Tractoriam.

Fig.1.

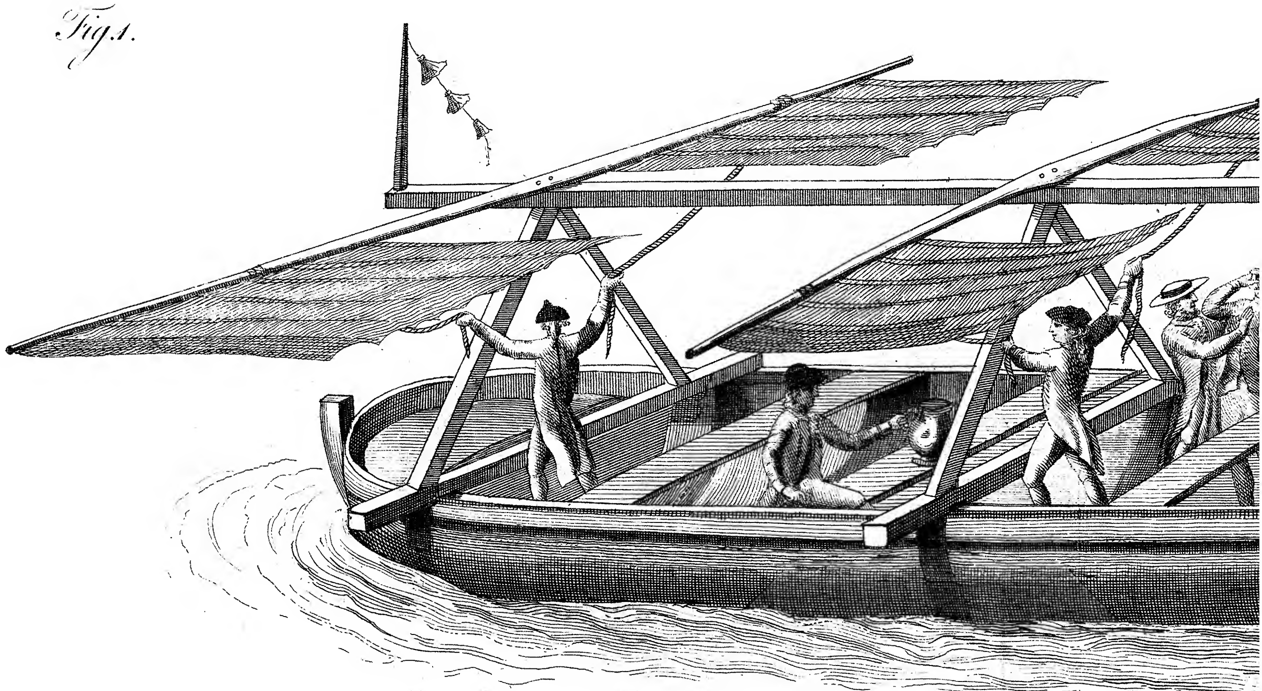
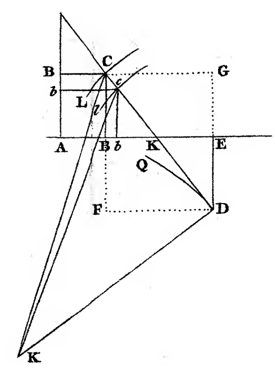
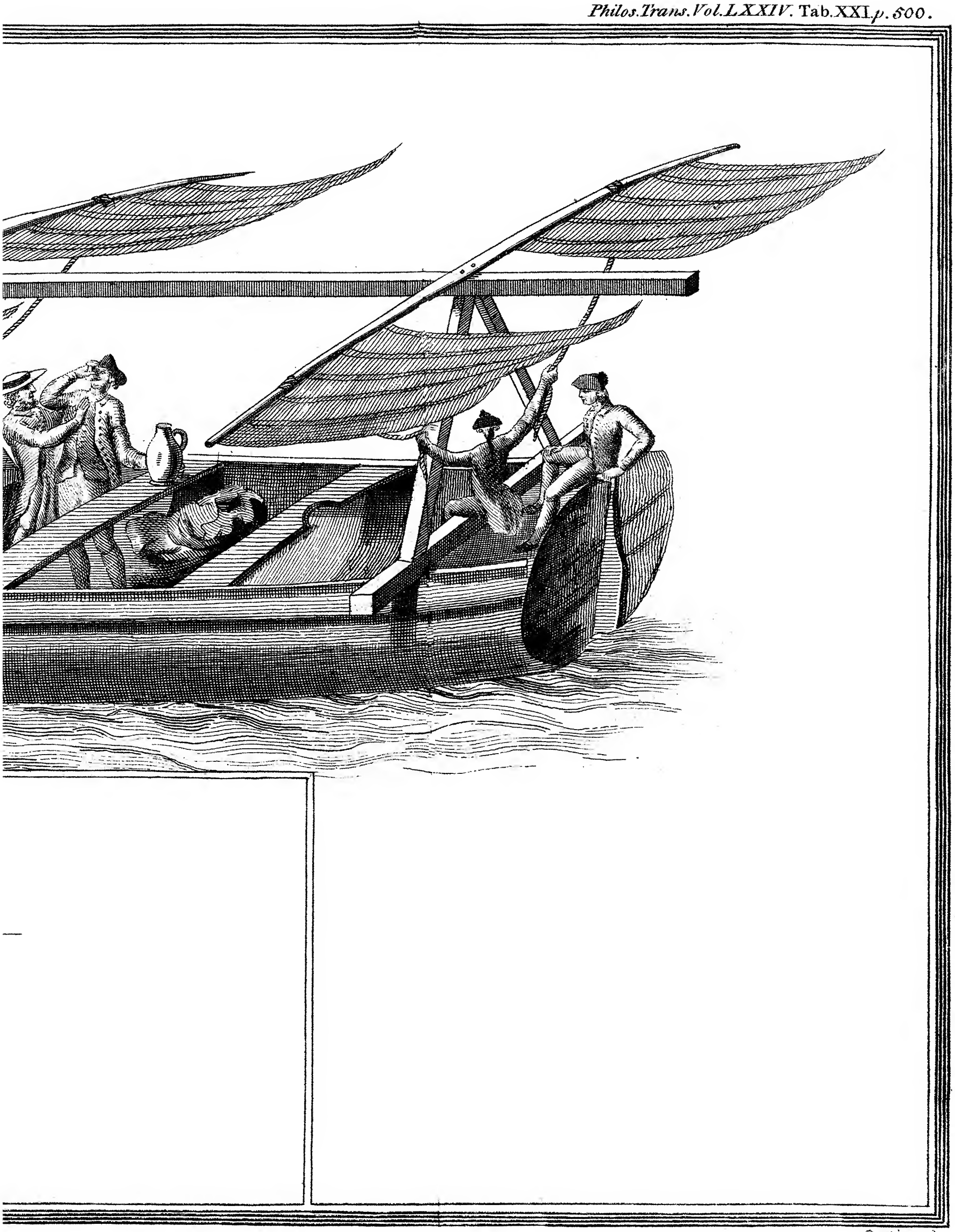


Fig.2.





Basire s.

Fig. 1

Fig. 2